Journal of Engineering Mathematics, Vol. 10, No. 1, January 1976
Noordhoff International Publishing–Leyden
Printed in The Netherlands

The existence of multiple solutions for the laminar flow in a uniformly porous channel with suction at both walls

W. A. ROBINSON

Department of Applied Mathematics and Theoretical Physics, University of Liverpool, Liverpool, England

(Received December 9, 1974 and in revised form May 27, 1975)

SUMMARY

A numerical investigation is made to establish whether multiple solutions exist for laminar, incompressible, steady flow in a parallel plate porous channel with uniform suction at both walls. For values of the wall suction Reynolds number, R, greater than 12.165 three numerical solutions are observed for each R, while for R less than 12.165 only one solution for each R can be found. A method involving the inclusion of exponentially small terms in a perturbation series is used to find two of the solutions analytically, while an appeal to the numerical results gives an indication of how the third solution can be obtained. The series involving the exponentially small terms, as well as predicting dual solutions, gives more accurate analytic results for the skin friction at the channel walls.

1. Introduction

Many research workers have investigated the steady, incompressible, laminar flow of fluid in channels and circular pipes with uniformly porous walls. Berman [1] showed that for constant suction or injection at the walls the solution of the flow equations in pipes and channels can be reduced to solving a single ordinary non-linear differential equation which involves a parameter R, the suction Reynolds number. Several series solutions can be obtained, for both pipes and channels, depending on whether R is positive (suction) or negative (injection), large or small.

The numerical investigation of the flow in a porous circular pipe by Terrill and Thomas [2] revealed dual solutions for all values of R outside the range $2.3 < R < 9.1$, but no solutions within this range. Moreover they found that the two solutions for large positive R, i.e. large suction, differed by exponentially small terms of the form $R^{-p}\,e^{-qR}$. Terrill [3] modified the single analytic solution for large R obtained in [2] to include exponentially small terms, and was then able to find the two solutions analytically.

Numerous numerical and analytic investigations by [4]–[7] and [1] on the flow through a uniformly porous channel have deduced that there is only one solution for each value of R, although solutions exist for the complete range of R, i.e. $-\infty < R < \infty$. Raithby [8] in a numerical investigation on the channel flow found that there was a second solution for values of R greater than 12, which changed its shape for $R > 27$. The numerical work in this paper has been concerned with finding multiple solutions for values of $R \geqq 0$, i.e. for suction at both walls, and has revealed that for $0 \leqq R < 12.165$ only one solution exists while in the range $12.165 < R < \infty$ three solutions exist for each value of R. The numerical results are discussed in chapter 3, where it can be seen that the difference between two of the solutions for large R exhibits the same characteristics as the difference between the dual solutions for large suction in pipe flow, due to exponentially small terms. Thus an analytic investigation similar to Terrill's [3] was carried out for large positive R in which exponentially small terms were included in the perturbation series solution. This produced dual analytic solutions for large suction, and a comparison of the numerical and analytic results can be found in chapter 6.

The interesting mathematical aspect of this problem, that could possibly be extended to other problems, is the way in which the inclusion of exponentially small terms leads to the prediction of dual solutions.

2. Formulation of the problem

Consider the steady, incompressible, laminar flow along a two-dimensional channel with porous walls through which fluid is injected or extracted with uniform speed V. Take x and y to be co-ordinate axes parallel and perpendicular to the channel walls, and assume u and v are the velocity components in the x and y directions respectively. Letting the channel width be $2h$ and introducing the dimensionless variable

$$\eta = y/h \tag{2.1}$$

reduces the Navier-Stokes equations to

$$u\frac{\partial u}{\partial x} + \frac{v}{h}\frac{\partial u}{\partial \eta} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2 u}{\partial \eta^2}\right), \tag{2.2a}$$

$$u\frac{\partial v}{\partial x} + \frac{v}{h}\frac{\partial v}{\partial \eta} = -\frac{1}{\rho h}\frac{\partial p}{\partial \eta} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2 v}{\partial \eta^2}\right), \tag{2.2b}$$

where ρ, p and ν are the density, pressure and kinematic viscosity of the fluid respectively.

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{1}{h}\frac{\partial v}{\partial \eta} = 0. \tag{2.3}$$

With the walls of the channel at $y = \pm h$, i.e. $\eta = \pm 1$, the boundary conditions are

$$\begin{aligned} &u(x, \pm 1) = 0, \qquad && v(x, 0) = 0, \\ &v(x, \pm 1) = \pm V, \qquad && \frac{\partial u}{\partial \eta}(x, 0) = 0. \end{aligned} \tag{2.4}$$

Berman [1] observed the equations of motion and the boundary conditions could be satisfied by assuming the velocity component v is independent of x and introducing a stream function, ψ, of the form

$$\psi = [hU(0) - Vx]f(\eta), \tag{2.5}$$

where $U(0)$ is an arbitrary velocity at $x = 0$.

With these assumptions the velocity components are given by

$$u = h^{-1}[hU(0) - Vx]f'(\eta), \tag{2.6a}$$

$$v = Vf(\eta). \tag{2.6b}$$

Substitution for u and v from (2.6) reduces the equations of motion to

$$\frac{\partial^2 p}{\partial x \partial \eta} = 0, \tag{2.7a}$$

$$f''' + R(f'^2 - ff'') = K, \tag{2.7b}$$

where $R = Vh/v$ is taken to be the suction Reynolds number of the flow and K is a constant, to be determined later as a function of R. Equation (2.7b) can be written

$$\varepsilon f''' + f'^2 - ff'' = \alpha^2, \tag{2.7c}$$

where $\varepsilon = 1/R$ and $\alpha^2 = K/R$.

The boundary conditions (2.4) become

$$\begin{aligned} f'(1) &= 0, \quad f(0) = 0, \\ f(1) &= 1, \quad f''(0) = 0. \end{aligned} \tag{2.8}$$

The condition $f(1) = 1$ implies that $R > 0$ for suction at both walls and $R < 0$ for injection at both walls.

3. Numerical solution of the equations of motion

The numerical solution of (2.7b), subject to boundary conditions (2.8), is a two-point boundary value problem. However, the equation can be solved with just one integration if the procedure outlined in chapter 5 of Terrill [4] is adopted.

This numerical investigation into the existence of multiple solutions was only carried out for R in the range $0 \leqq R < \infty$. The numerical results obtained are most clearly seen in Fig. 3.1, where $f''(1)$, which is proportional to the skin friction at the wall, is plotted against R for $R > 0$. In the range $12.165 < R < \infty$ triple solutions were found for each value of R, whilst only a single solution was observed for each R when $0 \leqq R < 12.165$.

The results of Fig. 3.1 will be discussed by dividing the figure into three sections, as follows:

1) Section I ($R = 0$ to $R = \infty$) covers the well behaved solutions for suction.

2) Section II ($R = 13.119$ to $R = \infty$) contains the solutions whose axial velocity profiles have a maximum located strictly between the centre of the channel and the wall but whose centreline velocity is positive.

3) Section III ($R = 13.119$ to $R = 12.165$ to $R = \infty$) includes axial velocity profiles with the same form as section II solutions but with reverse flow at the centre of the channel. This section contains dual solutions for the range $12.165 < R < 13.119$. These dual solutions have the same basic shape but differ in the value of the centreline velocity.

The deformation of the velocity profiles for section II and section III solutions is continuous, i.e. as $R \to 13.119$ from above and below the limiting profiles are identical.

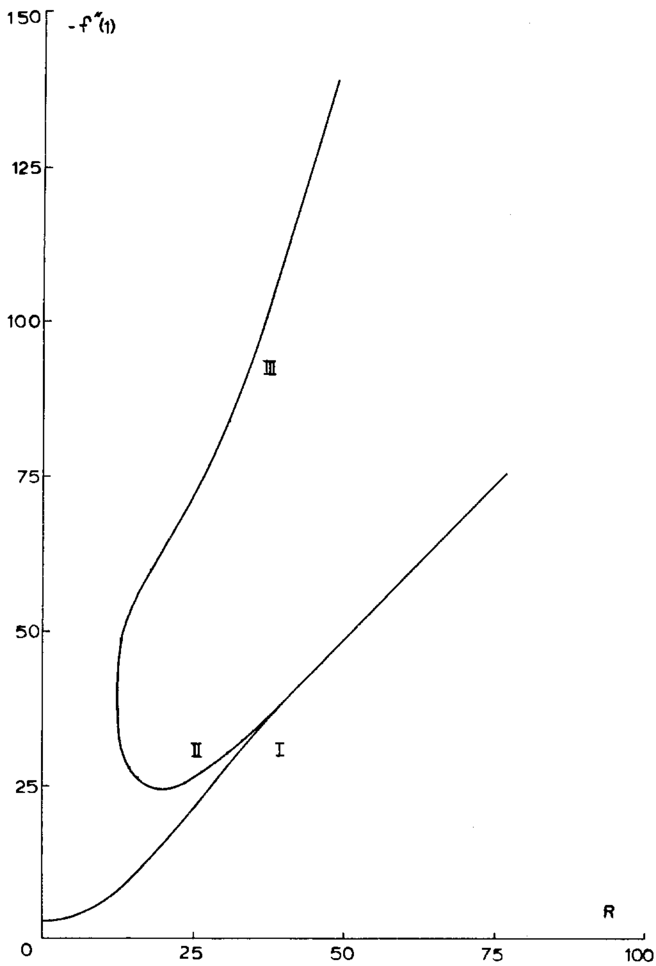

Figure 3.1. Variation of $f''(1)$ with R for values of R greater than zero.

The section I solutions agree with those obtained in previous papers, i.e. Terrill [4] and Berman [9]. They are characterized by a maximum value at the centreline which then decreases to zero at the wall with no turning points or points of inflexion in between. The deformation of the profiles in this range is from the parabolic form for $R = 0$, with a non-dimensional centreline velocity of 1.5, to the boundary layer form as $R \to \infty$ which has a non-dimensional velocity approximately equal to 1.0 everywhere except in the thin viscous layer near the wall.

Section II velocity profiles are given in Fig. 3.2. The non-dimensional velocity profiles for this range have a minimum at the centreline (which has a positive value) and then pass through a maximum before going to zero at the wall. This type of solution was first observed by Raithby [8] but only for values of R in the range $12 < R < 27$. Raithby said that for $R > 27$ the solution took the form that is familiar in pipe flow where the non-dimensional velocity has a maximum at the centreline and then passes through a minimum and another maximum before becoming zero at the wall. Such a change in the profile was not observed in this numerical survey, the solutions being of the same shape throughout the range. For

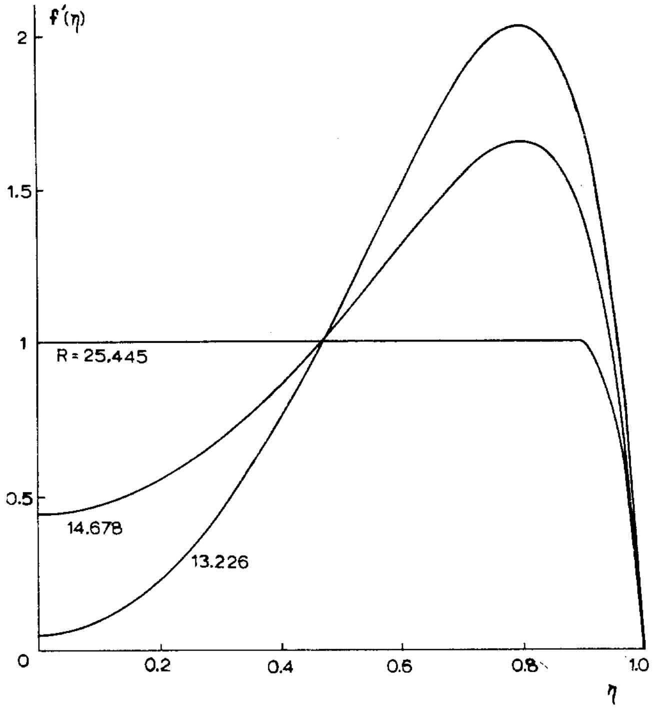

Figure 3.2. Axial velocity $f'(\eta)$ against non-dimensional channel distance η for section II wall suction.

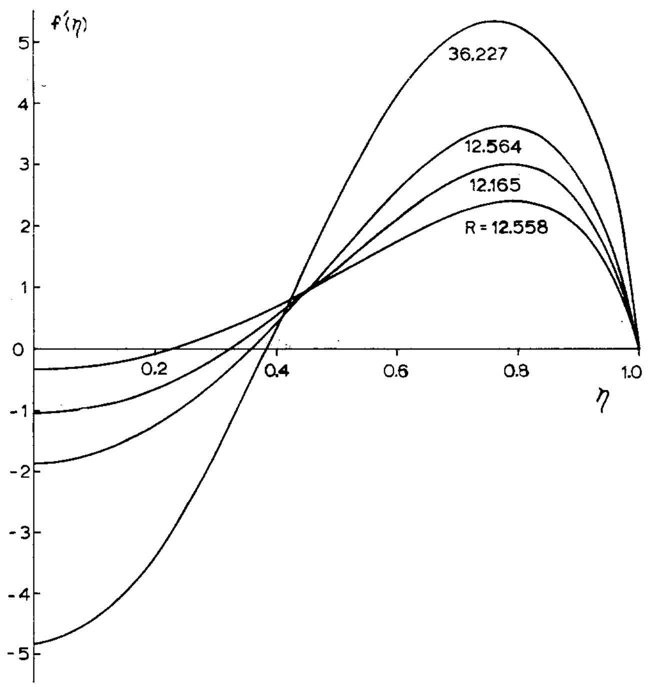

Figure 3.3. Axial velocity profiles for section III wall suction indicating the dual solutions in this range.

large R the values of the non-dimensional velocity at the minimum and at the maximum differ only by a small amount and are both approximately equal to 1.0. Furthermore the maximum occurs close to the wall, giving rise to a boundary layer type solution which does not differ much from section I large R solutions. In fact the difference in the solutions is found analytically by the introduction of exponentially small terms in chapter 5.

Fig. 3.3 illustrates the velocity profiles for section III solutions. These profiles have the same shape as those of section II except for a region of reverse flow around the centre of the channel. As R increases the point η_1, at which the maximum of $f'(\eta)$ occurs, becomes closer to the wall and the point η_2, at which $f'(\eta) = 0$, approaches the value $\eta_1/2$. Further, $f'(0)$, which is negative, decreases with increasing R while the value of $f'(0) + f'(\eta_1)$ tends to zero as $R \to \infty$.

The region between the maximum and the wall is small with the derivatives of $f(\eta)$ changing rapidly inside it and thus has the form of a viscous layer. From the numerical results this third solution for $f'(\eta)$ would appear to be a slight perturbation of a cosine inviscid solution, valid in the channel interior, matched to a viscous layer solution near the walls.

The results indicate that the solutions for large suction in each of the three sections consist of an outer inviscid solution, valid in the central region of the channel, and an inner solution, which is confined to the viscous layers near the walls. When R is large the velocity profiles for sections I and II are very similar, and in fact are exponentially small perturbations of the same solution. Thus, for the moment, a single analytic solution will be found to cover both of these sections and this will later be modified to deduce the two solutions of sections I and II. In this way we need only look for two complete solutions of (2.7c), one to cover the section III profiles and the other to act as a single solution for the dual solutions of sections I and II. A complete solution consists of an inner and an outer solution.

Both types of outer solution satisfy the inviscid equation

$$f'^2 - ff'' = \alpha^2, \tag{3.1}$$

and the outer boundary conditions

$$f(0) = 0, \quad f''(0) = 0. \tag{3.2}$$

Equation (3.1) has three different types of solution, linear, sinusoidal or hyperbolic. The particular types we require, corresponding to the two outer solutions, will be discussed in chapter 4.

Inside the viscous layer the complete equation (2.7c) must be solved, subject to the inner boundary conditions

$$f(1) = 0, \quad f'(1) = 1, \tag{3.3}$$

for the two separate cases.

To estimate the thickness of the viscous layer consider the transformation

$$f(\eta) = g(\zeta), \tag{3.4}$$

where

$$\zeta = a(1 - \eta), \tag{3.5}$$

and a is a function of R.

Substitution into (2.7c) yields that $g(\zeta)$ satisfies

$$\varepsilon a^3 g''' - a^2(g'^2 - gg'') = -\alpha^2. \tag{3.6}$$

The viscous and inertia terms are of equal order if

$$\varepsilon a^3 = a^2, \tag{3.7}$$

which gives $a = R$.

Thus both types have a viscous layer of order $1/R$, even though the two inner solutions are different. The two inner solutions will have different values for the constant α, and these will be investigated in chapter 4.

4. Analytic solutions for sections I and II and section III profiles

In this chapter the outer solution for section III profiles and an outer and inner solution, with exponentially small terms neglected, for sections I and II are determined. As remarked previously, both the outer solutions require (3.1) to be solved subject to (3.2).

4.1. *Solution for section III*

To determine the outer solution first differentiate (3.1) to obtain

$$f'f'' - ff''' = 0, \tag{4.1}$$

which, if $f'' \not\equiv 0$, may also be written

$$\frac{f'}{f} = \frac{f'''}{f''}. \tag{4.2}$$

Integrating (4.2) gives

$$f'' = C^2 f, \tag{4.3}$$

where C^2 is a constant, positive or negative. The solutions of (4.3), for which $f''(\eta) \not\equiv 0$, are

$$f(\eta) = A\sin(C\eta + B), \tag{4.4a}$$

$$f(\eta) = A\sinh(C\eta + B), \tag{4.4b}$$

where A and B are constants and C is real. The boundary conditions give

$$B = 0. \tag{4.5}$$

The numerical solutions for $f'(\eta)$ indicate that

$$f'(\eta) = D\cos \pi\eta, \tag{4.6}$$

which corresponds to the analytic solution, (4.4a) with (4.5), when $C = \pi$ and $CA = D$.

When (4.6) is substituted into (3.1) we obtain

$$D^2 = \alpha^2$$

and thus

$$D = \alpha. \tag{4.7}$$

The outer solution can therefore be written as

$$f'(\eta) = \alpha \cos \pi\eta, \tag{4.8}$$

i.e.

$$f(\eta) = \frac{\alpha}{\pi} \sin \pi\eta. \tag{4.9}$$

From (4.8) we note that

$$\alpha = f'(0). \tag{4.10}$$

TABLE 4.1

Values of η and $f'(\eta)$ at some important points in the section III large R solutions

R	η_2	η_1	$f'(\eta_1)$	$f'(0)$	$(1-\eta_1)R$	$f'(0)/R$
118.73	0.4179	0.8328	6.755	− 6.635	19.8	−5.588, −2
367.72	0.4588	0.9171	12.691	−12.659	30.5	−3.443, −2
449.92	0.4643	0.9283	14.597	−14.572	32.3	−3.239, −2
574.96	0.4702	0.9403	17.420	−17.401	34.3	−3.026, −2
680.58	0.4738	0.9474	19.743	−19.726	35.8	−2.898, −2
787.33	0.4766	0.9530	22.040	−22.026	37.0	−2.798, −2
895.10	0.4788	0.9575	24.316	−24.304	38.0	−2.715, −2
1003.78	0.4806	0.9611	26.572	−26.562	39.1	−2.646, −2

Some features of the section III large R solutions are shown in Table 4.1. The value of $(1 - \eta_1)R$ can be seen to be almost a constant, verifying that $(1 - \eta_1)$, which is a measure of the thickness of the viscous layer, is of order $1/R$. Furthermore, the results in the table indicate that α $(= f'(0))$ is proportional to R. This can be deduced from equation (3.6), as $\alpha^2 = a^2 = R^2$ for the constant term of the inner equation to be of the same order as the viscous and inertia terms. The difference in the two inner solutions illustrates why α is of different order for the two cases, a point that will later be verified by showing that $\alpha = 0(1)$ for sections I and II. An investigation of the numerical results for large R indicates that none of the individual terms of (2.7c) are completely dominant throughout the viscous layer. Thus no simplification of the equation, that would be valid throughout the layer, could be made and the full equation would have to be solved. As this solution, with its reverse flow at the channel centre, seems physically unreasonable no effect was made to obtain an analytic solution in the viscous layer.

The first three terms of an approximate series expansion for α can be determined from the numerical results for large R. The series

$$\frac{f'(0)}{R} = -0.02 - \frac{7.9}{R} + \frac{1450}{R^2} \tag{4.11}$$

gives results accurate to at least three decimal places when $R > 600$.

4.2. *Solution for sections I and II neglecting exponentially small terms*

4.2.1. *Outer solution*

One would assume that exerting large suction on the flow in a channel would produce boundary layers at the walls. Since the flow for these two sections should be fairly well behaved it appears physically reasonable to expect an outer solution of the form $f(\eta) \sim \eta$, and this is indeed substantiated by the numerical results.

As stated previously, the inviscid equation (3.1) has a linear solution

$$f(\eta) = A\eta + B, \tag{4.12}$$

corresponding to the case $f''(\eta) \equiv 0$, and clearly this is the appropriate choice of outer solution. This solution satisfies the boundary conditions and the equation of motion if $A = \alpha$ and $B = 0$, and then becomes

$$f(\eta) = \alpha\eta. \tag{4.13}$$

The normal wall condition, $f(1) = 1$, suggests $\alpha = 0(1)$, and this is confirmed by the numerical results. We write α as

$$\alpha = \sum_{r=0}^{\infty} \alpha_r \varepsilon^r, \tag{4.14}$$

where the coefficients α_r are constants determined by matching (4.13) with the inner solution.

4.2.2. *Inner solution*

As the viscous layer is of order R^{-1} we use the stretching transformation

$$(1 - \eta) = \varepsilon t. \tag{4.15}$$

With this stretch the inner boundary conditions (3.3) become

$$f(0) = 1, \quad f'(0) = 0, \tag{4.16}$$

and (2.7c) takes the form

$$f''' + ff'' - f'^2 = -\alpha^2\varepsilon^2, \tag{4.17}$$

where primes denote differentiation with respect to t.

We note that as $\alpha = 0(1)$ the right hand side of (4.17) is of $0(\varepsilon^2)$, whereas for the section III case it would be of order 1, and thus the constant term will not be as significant in the viscous layer for this case.

Since the inner solution must satisfy the condition $f(0) = 1$, a perturbation solution of the form

$$f(t) = 1 + \sum_{r=1}^{\infty} f_r(t)\varepsilon^r \tag{4.18}$$

is sought. The boundary conditions to be satisfied by $f_r(t)$ are, therefore, from (4.16)

$$f_r(0) = 0, \quad f_r'(0) = 0. \tag{4.19}$$

Substituting (4.14) and (4.18) into (4.17) and equating coefficients of ε^r yields

$$f_1''' + f_1'' = 0, \tag{4.20}$$

$$f_2''' + f_2'' + f_1 f_1'' - (f_1')^2 = -\alpha_0^2, \tag{4.21}$$

. . . .

The solution of (4.20) subject to (4.19) is

$$f_1(t) = A_1(t - 1 + e^{-t}), \tag{4.22}$$

where A_1 is an arbitrary constant. Similarly, as the equation for each $f_r(t)$ is of third order and there are only two boundary conditions each will contain an arbitrary constant A_r, say. For simplicity, the constants A_r and the coefficients α_r of α will be obtained at each iteration by matching the inner and outer solutions.

The first two terms of the inner expansion are

$$f(t) = 1 + A_1\varepsilon(t - 1 + e^{-t}). \tag{4.23}$$

The outer solution (4.13) expressed in terms of the inner variable t is

$$f(t) = \alpha_0 + (\alpha_1 - \alpha_0 t)\varepsilon + (\alpha_2 - \alpha_1 t)\varepsilon^2 + \dots . \tag{4.24}$$

Matching the inner solution (4.23), as $t \to \infty$, with (4.24) gives

$$\begin{aligned} &\alpha_0 = 1, \\ &A_1(t - 1) = \alpha_1 - t, \end{aligned} \tag{4.25}$$

and thus

$$\alpha_1 = -A_1 = 1. \tag{4.26}$$

Solving the equations for f_2, f_3 and f_4 in a similar manner gives

$$\begin{aligned} f_1(t) &= 1 - t - e^{-t}, \\ f_2(t) &= 4 - t - \left(\frac{t^2}{2} + 3t + 4\right)e^{-t}, \\ f_3(t) &= \tfrac{129}{4} - 4t - \left(\frac{t^4}{8} + \tfrac{3}{2}t^3 + 9t^2 + 28t + \tfrac{65}{2}\right)e^{-t} + \frac{e^{-2t}}{4}, \\ f_4(t) &= \tfrac{26377}{72} - \tfrac{129}{4}t - \left(\frac{t^6}{48} + \tfrac{3}{8}t^5 + 4t^4 + \tfrac{82}{3}t^3 + \tfrac{497}{4}t^2 \right. \\ &\quad \left. + \tfrac{1317}{4}t + \tfrac{8963}{24}\right)e^{-t} + \left(\frac{t^2}{4} + \tfrac{9}{4}t + \tfrac{57}{8}\right)e^{-2t} - \frac{e^{-3t}}{72}. \end{aligned} \tag{4.27}$$

The coefficients α_r are given by

$$\alpha_0 = 1,\ \alpha_1 = 1,\ \alpha_2 = 4,\ \alpha_3 = \tfrac{129}{4},\ \alpha_4 = \tfrac{26377}{72}. \tag{4.28}$$

5. Solution for sections I and II including exponentially small terms

The solution of the previous chapter agrees fairly accurately with both the section I and II numerical solutions for large R. An effort to correctly predict these two solutions will now be made by investigating the terms neglected in the matching process. The terms of the form $t^r\,e^{-nt}$ in the inner expansion (4.27) have not been matched with any corresponding exponentially small terms in the outer expansion. In this chapter the outer expansion will be modified to take into account these exponentially small terms.

5.1. *Modified outer solution*

To obtain the modified outer solution the complete equation, i.e. (2.7c), is solved in the outer region subject to the outer conditions (3.2).

If the modified outer solution is assumed to be

$$f(\eta) = \alpha\eta + \gamma F_1(\eta) + \gamma^2 F_2(\eta) + \ldots, \tag{5.1}$$

where γ is exponentially small, then substitution in (2.7c) yields that $F_1(\eta)$ satisfies

$$\varepsilon F_1''' - \alpha\eta F_1'' + 2\alpha F_1' = 0. \tag{5.2}$$

The outer conditions (3.2), in terms of $F_r(\eta)$, become

$$F_r(0) = 0, \quad F_r''(0) = 0. \tag{5.3}$$

Differentiate (5.2) to obtain

$$\varepsilon F_1^{\text{iv}} - \alpha\eta F_1''' + \alpha F_1'' = 0, \tag{5.4}$$

from which we deduce

$$F_1^{\text{iv}}(0) = 0. \tag{5.5}$$

An extra differentiation of (5.4) gives

$$\varepsilon F_1^{\text{v}} - \alpha\eta F_1^{\text{iv}} = 0, \tag{5.6}$$

whose solution is

$$F_1^{\text{iv}}(\eta) = C\exp(\alpha\eta^2/2\varepsilon), \tag{5.7}$$

where C is a constant. Equation (5.5) implies $C = 0$ and thus

$$F_1(\eta) = a\eta^3 + b\eta^2 + d\eta + e, \tag{5.8}$$

where a, b, d and e are constants. The boundary conditions give $b = e = 0$, and substitution of (5.8) into (5.2) yields that $\alpha d = -3\varepsilon a$. Thus a solution for $F_1(\eta)$ is

$$F_1(\eta) = \eta^3 + d\eta, \tag{5.9}$$

where $d = -3\varepsilon/\alpha$. (5.10)

The equation for $F_2(\eta)$ is

$$\varepsilon F_2''' - \alpha\eta F_2'' + 2\alpha F_2' = F_1 F_1'' - F_1'^2$$
$$= -3\eta^4 - d^2. \tag{5.11}$$

Differentiating (5.11) twice with respect to η gives

$$\varepsilon F_2^{\mathrm{v}} - \alpha\eta F_2^{\mathrm{iv}} = -36\eta^2. \tag{5.12}$$

The solution of (5.12) subject to the condition $F_2^{\mathrm{iv}}(0) = 0$, obtained by the differentiation of (5.11) and substitution from (5.3), is

$$\begin{aligned} F_2^{\mathrm{iv}}(\eta) &= -\frac{36}{\varepsilon}\exp(\alpha\eta^2/2\varepsilon)\int_0^{\eta} s^2 \exp(-\alpha s^2/2\varepsilon)\,ds \\ &= \frac{36}{\alpha}\eta - \frac{36}{\alpha}\exp(\alpha\eta^2/2\varepsilon)\int_0^{\eta}\exp(-\alpha s^2/2\varepsilon)\,ds \\ &= \frac{36}{\alpha}\eta - \frac{36}{\alpha}\left(\frac{\pi\varepsilon}{2\alpha}\right)^{\frac{1}{2}}\exp(\alpha\eta^2/2\varepsilon)\,\mathrm{erf}(\eta/\sqrt{2\varepsilon/\alpha}) \\ &= \frac{36}{\alpha}\eta - \frac{36}{\alpha}\left[\left(\frac{\pi\varepsilon}{2\alpha}\right)^{\frac{1}{2}}\exp(\alpha\eta^2/2\varepsilon)\right. \\ &\qquad \left. - \frac{\varepsilon}{\alpha\eta}\left\{1 + \sum_{m=1}^{M}(-1)^m \frac{1\cdot 3\cdot \ldots \cdot(2m-1)}{(\alpha\eta^2/\varepsilon)^m} + 0(|\eta R^{\frac{1}{2}}|^{-2M+1})\right\}\right], \end{aligned} \tag{5.13}$$

where the asymptotic approximation of the error function has been used.

When expressed in terms of t the function $F_1(\eta)$ is a simple polynomial and, therefore, is not the term to be matched with the exponential terms in (4.27). Equation (5.13) contains exponentials and thus to see if it will match with the terms of the form $t^r\,\mathrm{e}^{-t}$ in (4.27) its behaviour for large R when $\eta \to 1$ will be examined. From (5.13), as $R \to \infty$ and $\eta \to 1$

$$F_2^{\mathrm{iv}}(\eta) \sim \frac{-36}{\alpha}\left(\frac{\pi\varepsilon}{2\alpha}\right)^{\frac{1}{2}}\exp(\alpha\eta^2/2\varepsilon),$$

which in terms of t

$$\begin{aligned} &= -36\left(\frac{\pi\varepsilon}{2\alpha^3}\right)^{\frac{1}{2}}\exp\{(1 - 2\varepsilon t + \varepsilon^2 t^2)\alpha/2\varepsilon\} \\ &= -36\left(\frac{\pi\varepsilon}{2\alpha^3}\right)^{\frac{1}{2}}\exp\left(\frac{\alpha t^2\varepsilon}{2}\right)\exp\{(1-\alpha)t\}\exp\left(\frac{\alpha}{2\varepsilon}\right)\exp(-t). \end{aligned} \tag{5.14}$$

Equation (5.14) contains e^{-t}, so clearly F_2 will match with those terms of the form $t^r\,\mathrm{e}^{-t}$ in (4.27). The matching will be carried out using the fourth derivative of those terms in equation (4.27) which involve e^{-t}, that is

$$\begin{aligned} \frac{d^4 f}{d\eta^4} &= \varepsilon^{-4}\frac{d^4 f}{dt^4} = -\varepsilon^{-3}\,\mathrm{e}^{-t}\left\{1 + \varepsilon\left(\frac{t^2}{2} - t - 2\right) + \varepsilon^2\left(\frac{t^4}{8} - \frac{t^3}{2} - 2t - \frac{9}{2}\right)\right. \\ &\quad \left. + \varepsilon^3\left(\frac{t^6}{48} - \frac{t^5}{8} + \frac{t^4}{4} - \frac{5t^3}{3} + \frac{7t^2}{4} - \frac{79t}{4} - \frac{301}{24}\right) + \ldots\right\}. \end{aligned} \tag{5.15}$$

In (5.14) the first two exponentials are developed in a series expansion in t. This expansion is the same as the expansion in (5.15) apart from a constant factor, which is a function of ε only. It is the appearance of this constant factor that allows us to determine the exponentially small parameter γ^2 in terms of ε. If γ^2 was known and the expression $\gamma^2 \times$ equation (5.14) evaluated as a power series in t, then all the powers of t in this expansion would match identically with the corresponding powers of t in equation (5.15). Matching (5.14) and (5.15) yields

$$36\left(\frac{\pi\varepsilon^7}{2\alpha^3}\right)^{\frac{1}{2}}\gamma^2 \exp\left(\frac{\alpha}{2\varepsilon}\right) = \left\{1 + \varepsilon\left(\frac{t^2}{2} - t - 2\right) + \varepsilon^2\left(\frac{t^4}{8} - \frac{t^3}{2} - 2t - \tfrac{9}{2}\right)\right.$$
$$\left. + \varepsilon^3\left(\frac{t^6}{48} - \frac{t^5}{8} + \frac{t^4}{4} - \frac{5t^3}{3} + \frac{7t^2}{4} - \frac{79t}{4} - \frac{301}{24}\right) + \ldots\right\}$$
$$\times \exp\{(\alpha - 1)t\} \exp(-\alpha t^2\varepsilon/2). \tag{5.16}$$

When the right hand side of equation (5.16) is evaluated the terms in t and higher powers cancel each other out leaving

$$36\left(\frac{\pi\varepsilon^7}{2\alpha^3}\right)^{\frac{1}{2}}\gamma^2 \exp\left(\frac{\alpha}{2\varepsilon}\right) = 1 - 2\varepsilon - \tfrac{9}{2}\varepsilon^2 - \tfrac{301}{24}\varepsilon^3 + 0(\varepsilon^4), \tag{5.17}$$

i.e.

$$\gamma^2 = \frac{\alpha^2}{36}\left(\frac{2}{\pi\varepsilon^7}\right)^{\frac{1}{2}} \exp\left\{-\frac{(1+\varepsilon^{-1})}{2}\right\}\left\{1 - \frac{9\varepsilon}{2} - \frac{57\varepsilon^2}{4} - \frac{18313\varepsilon^3}{144} + 0(\varepsilon^4)\right\}. \tag{5.18}$$

Clearly, for large R, equation (5.18) will give two roots for γ, and these will correspond to the dual solutions.

Equation (5.18) will yield two roots for γ if the series on the right hand side is positive. To find the critical value of ε, and hence R, at which two roots first exist the zero of the above series must be calculated. In its present form equation (5.18) may not provide a good approximation to this critical value of ε. This is because the coefficients of ε^r in the series all have the same sign and are increasing in size with increasing r. These facts infer that the term of $0(\varepsilon^4)$ could be significant. The right hand side of (5.18) can be modified by multiplying the series by powers of α^{-2} to produce an alternative series with better convergence. This procedure can change the coefficients of ε^r quite drastically, but it was found that probably the most accurate series is obtained when γ^2 is written as

$$\gamma^2 = \frac{\alpha^8}{36}\left(\frac{2}{\pi\varepsilon^7}\right)^{\frac{1}{2}} \exp\left\{-\frac{(1+\varepsilon^{-1})}{2}\right\}\left\{1 - \frac{21\varepsilon}{2} + \frac{39\varepsilon^2}{4} - \frac{15793\varepsilon^3}{144} + 0(\varepsilon^4)\right\}. \tag{5.19}$$

If the terms of $0(\varepsilon^4)$ are now considered negligible in (5.19) then there are two roots for γ, corresponding to the dual solutions, provided $\varepsilon < 0.0947$, i.e., provided $R > 10.6$. The figure 10.6 as the value of R at which dual solutions begin was derived by the use of a method that has its greatest accuracy when R is large, which may account for the difference between it and the numerical result of 12.165.

Expanding (5.19) for $R > 10.6$ gives

$$\gamma = \pm \frac{1}{6}\left(\frac{2}{\pi\varepsilon^7}\right)^{\frac{1}{4}} \exp\left\{-\frac{(1+\varepsilon^{-1})}{4}\right\}\{1 - \tfrac{5}{4}\varepsilon - \tfrac{253}{32}\varepsilon^2 - \tfrac{82621}{1152}\varepsilon^3 + 0(\varepsilon^4)\}. \tag{5.20}$$

A comparison of the accuracy of γ as given by (5.20) with the numerical results will be made in chapter 6.

5.2. *Modified inner solution*

The term $\gamma F_1(\eta)$ introduced into the modified outer expansion was not matched with any terms in the inner expansion. Consequently the inner solution must be modified to include terms which match with $\gamma F_1(\eta)$, and will therefore be written in the form

$$f(t) = 1 + \sum_{r=1}^{\infty} f_r(t)\varepsilon^r + \gamma_1 \sum_{r=0}^{\infty} h_r(t)\varepsilon^r + 0(\gamma_1^2). \tag{5.21}$$

Also, as there are dual solutions the constant α will take two values, depending on the sign of γ, and will therefore be written

$$\alpha = \sum_{r=0}^{\infty} \alpha_r\varepsilon^r + \gamma \sum_{r=0}^{\infty} \beta_r\varepsilon^r + 0(\gamma^2). \tag{5.22}$$

In (5.21) and (5.22) β_r and $h_r(t)$ must satisfy equation (4.17) and the boundary conditions (4.16), while γ_1 is an exponentially small term related to γ. The wall conditions (4.16) for $h_r(t)$ are

$$h_r(0) = h_r'(0) = 0. \tag{5.23}$$

The first two terms of the modified outer solution (5.1), when (5.9), (5.10) and (5.22) are used, become

$$f(\eta) = \sum_{r=0}^{\infty} \alpha_r\varepsilon^r\eta + \gamma \sum_{r=0}^{\infty} \beta_r\varepsilon^r\eta + \gamma\left(\eta^3 - \frac{3\varepsilon\eta}{\alpha}\right) + \dots . \tag{5.24}$$

The terms of order γ in the above are

$$\gamma\left\{\eta^3 + \left(\sum_{r=0}^{\infty} \beta_r\varepsilon^r - \frac{3\varepsilon}{\alpha}\right)\eta\right\}. \tag{5.25}$$

As α^{-1} can be expanded as a power series in ε the coefficient of η in (5.25) can be reduced to the single term

$$\sum_{r=0}^{\infty} \beta_r\varepsilon^r - \frac{3\varepsilon}{\alpha} = \omega = \sum_{r=0}^{\infty} \omega_r\varepsilon^r, \tag{5.26}$$

which results in a much simpler and clearer matching of the modified outer and inner solutions. In terms of the inner variable t and the constant ω, equation (5.25) becomes

$$\gamma\{1 + \omega_0 + (\omega_1 - \omega_0 t - 3t)\varepsilon + (\omega_2 - \omega_1 t + 3t^2)\varepsilon^2 + (\omega_3 - \omega_2 t - t^3)\varepsilon^3 + (\omega_4 - \omega_3 t)\varepsilon^4 + \dots\}. \tag{5.27}$$

The term $\gamma_1 \sum_{r=0}^{\infty} h_r(t)\varepsilon^r$ will now be matched with the terms of $0(\gamma)$ in (5.24), i.e. with equation (5.27), and will thus allow the coefficients ω_r and the exponentially small term γ_1 to be determined. The coefficients β_r of α can be recovered from the ω_r's by the use of (5.26) and (4.28).

As α appears in the inner equation (4.17) the coefficients β_r will be introduced into the differential equations for the $h_r(t)$. This will require the recovery of β_r from ω_r at each step in the evaluation of the $h_r(t)$'s. This is a cumbersome method which need not be employed if the differentiated form of the inner equation, that is

$$f^{\text{iv}} + ff''' - f'f'' = 0, \tag{5.28}$$

is used.

Substitution of (5.21) into (5.28), and equating coefficients of γ_1, gives that the equation for $h_0(t)$ is

$$h_0^{\text{iv}} + h_0''' = 0. \tag{5.29}$$

The solution of (5.29) that satisfies the boundary conditions (5.23) is

$$h_0(t) = A_0(t - 1 + e^{-t}) + B_0 t^2, \tag{5.30}$$

where A_0 and B_0 are constants. If we let $t \to \infty$ in (5.30) and match with (5.27) we get

$$A_0\gamma_1(t - 1) + B_0\gamma_1 t^2 \sim \gamma\{1 + \omega_0 + (\omega_1 - \omega_0 t - 3t)\varepsilon + \ldots\}. \tag{5.31}$$

The terms of $0(\gamma)$ match only if

$$1 + \omega_0 = 0,$$

i.e.

$$\omega_0 = -1. \tag{5.32}$$

The term of $0(\varepsilon\gamma)$ on the right hand side of (5.31) indicates that

$$\gamma_1 = \gamma\varepsilon \tag{5.33}$$

and therefore

$$A_0 = -\omega_0 - 3 = -2, \qquad \omega_1 = -A_0 = 2, \qquad B_0 = 0, \tag{5.34}$$

which give

$$h_0(t) = -2(t - 1 + e^{-t}). \tag{5.35}$$

The coefficient of $\varepsilon\gamma_1$ gives that $h_1(t)$ satisfies

$$\begin{aligned} h_1^{\text{iv}} + h_1''' &= h_0' f_1'' + h_0'' f_1' - h_0 f_1''' - h_0''' f_1 \\ &= 4t\, e^{-t}. \end{aligned} \tag{5.36}$$

The solution of (5.36) subject to the conditions at $t = 0$ is

$$\begin{aligned} h_1(t) = {} & B_1 t^2 + (A_1 - 2B_1)t + (12 - A_1 + 2B_1) \\ & - (2t^2 + 12t + 12 - A_1 + 2B_1)\, e^{-t}. \end{aligned} \tag{5.37}$$

The coefficient of $\gamma\varepsilon^2$ in the matching of (5.37), for large t, with (5.27) yields

$$B_1 = 3, \; A_1 = 4, \; \omega_2 = 14, \tag{5.38}$$

and therefore

$$h_1(t) = 3t^2 - 2t + 14 - (2t^2 + 12t + 14)\,e^{-t}. \tag{5.39}$$

The equation for $h_2(t)$ is

$$\begin{aligned} h_2^{iv} + h_2''' &= h_0'f_2'' + h_0''f_2' - h_0f_2''' - h_0'''f_2 + h_1'f_1'' + h_1''f_1' - h_1f_1''' - h_1'''f_1 \\ &= (3t^3 - 3t^2 - 10t - 6)\,e^{-t} + 12e^{-2t} - 6. \end{aligned} \tag{5.40}$$

We solve for $h_2(t)$ in a similar manner to get

$$h_2(t) = \tfrac{247}{2} - 14t - t^3 - (\tfrac{3}{4}t^4 + 8t^3 + 40t^2 + 108t + 125)\,e^{-t} + \tfrac{3}{2}e^{-2t}, \tag{5.41}$$

and $\omega_3 = \frac{247}{2}$.

Calculating the coefficients β_r of α, using equation (5.26), yields

$$\beta_0 = -1, \; \beta_1 = 5, \; \beta_2 = 11, \; \beta_3 = \tfrac{229}{2}. \tag{5.42}$$

6. Comparison of the numerical and the modified analytic solutions

A check on the accuracy of the modified solution will now be made by comparing two of its main features with numerical results.

For the ordinary outer solution (4.13), $f'''(\eta)$ at $\eta = 0$ is identically zero, although this is not the case in the numerical analysis. By use of the modified outer solution (5.1) we see that

$$\begin{aligned} [f'''(\eta)]_{\eta=0} &= 6\gamma + 0(\gamma^2) \\ &\sim \pm\left(\frac{2}{\pi\varepsilon^7}\right)^{\frac{1}{2}} \exp\left\{-\frac{(1+\varepsilon^{-1})}{4}\right\}\{1 - \tfrac{5}{4}\varepsilon - \tfrac{253}{32}\varepsilon^2 - \tfrac{82621}{1152}\varepsilon^3 + 0(\varepsilon^4)\}, \end{aligned}$$

which, in terms of R,

$$= \pm\left(\frac{2R^7}{\pi}\right)^{\frac{1}{2}} \exp\left\{-\frac{(1+R)}{4}\right\}\left\{1 - \frac{5}{4R} - \frac{253}{32R^2} - \frac{82621}{1152R^3} + 0\left(\frac{1}{R^4}\right)\right\}. \tag{6.1}$$

In Table 6.1 numerical results for $f'''(\eta)$ at $\eta = 0$ are compared with the analytic values obtained by the use of the first one, two, three and four terms of equation (6.1). These values, when first observed, appear to be more accurate for the section I solutions, as the section I results converge to the numerical results when the number of terms is increased whereas the section II values are closer to the numerical results when only two or three terms are used. However, the section II results are more accurate, although the error, as the number of terms included is increased, tends to the same value for both section I and II results and corresponds to the term of $0(\gamma^2)$ omitted.

In previous papers on flow in porous channels and pipes the main check on the accuracy of the analytic solutions has come through the evaluation of $f''(\eta)$ at $\eta = 1$, which is proportional to the axial skin friction at the wall. Prior to the inclusion of exponentially

TABLE 6.1

Numerical values of $f'''(\eta)$ at $\eta = 0$, for various R in different sections, compared with the analytical formula (6.1)

R	*Section*	*Analytical – Number of terms of* (6.1) *used*				*Numerical*
		First	*First two*	*First three*	*First four*	
28.142	I	−2.105, −1	−2.012, −1	−1.991, −1	−1.984, −1	−1.790, −1
37.519	I	−3.340, −2	−3.229, −2	−3.210, −2	−3.205, −2	−3.128, −2
49.098	I	−2.958, −3	−2.883, −3	−2.873, −3	−2.871, −3	−2.862, −3
55.934	I	−6.728, −4	−6.578, −4	−6.561, −4	−6.558, −4	−6.552, −4
65.674	I	−7.805, −5	−7.656, −5	−7.642, −5	−7.640, −5	−7.639, −5
28.139	II	2.106, −1	2.013, −1	1.992, −1	1.985, −1	2.240, −1
37.791	II	3.160, −2	3.056, −2	3.038, −2	3.034, −2	3.104, −2
44.717	II	7.510, −3	7.301, −3	7.271, −3	7.265, −3	7.310, −3
56.008	II	6.620, −4	6.472, −4	6.455, −4	6.453, −4	6.457, −4
64.296	II	10.613, −5	10.406, −5	10.386, −5	10.383, −5	10.384, −5

small terms in the series, the analytic formula used for the comparison of $f''(\eta)$ at $\eta = 1$ with the numerical results was that obtained from the inner solution (4.18) with (4.27), i.e.

$$[f''(\eta)]_{\eta=1} = \varepsilon^{-2}[f''(t)]_{t=0} = -\left\{\frac{1}{\varepsilon} - 1 - \tfrac{13}{2}\varepsilon - \tfrac{677}{12}\varepsilon^2 + 0(\varepsilon^3)\right\},$$

which, in terms of R,

$$= -\left\{R - 1 - \frac{13}{2R} - \frac{677}{12R^2} + 0\left(\frac{1}{R^3}\right)\right\}. \tag{6.2}$$

The two results required for $f''(\eta)$ at $\eta = 1$ are obtained from the modified inner solution (5.21), which is a combination of (6.2) and a correction due to the exponentially small terms. This correction is

$$\frac{\gamma\varepsilon}{\varepsilon^2}\{h_0''(0) + \varepsilon h_1''(0) + \varepsilon^2 h_2''(0) + \dots\} + 0(\gamma^2),$$

which

$$= \pm\frac{1}{6}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} R^{11/4}\exp\left\{-\frac{(1+R)}{4}\right\}\left\{1 - \frac{5}{4R} - \frac{253}{32R^2} - \frac{82621}{1152R^3} + \dots\right\}$$
$$\times\left\{-2 + \frac{12}{R} + \frac{17}{R^2} + \dots\right\} + 0(\gamma^2). \tag{6.3}$$

Table 6.2 compares the numerical results for $f''(\eta)$ at $\eta = 1$ with the analytic results, which include the exponentially small correction. For any R greater than about 45 the results in either section have the same error, which is due to the terms of $0(R^{-3})$ omitted from (6.2). In the middle range $30 < R < 45$ the inaccuracy in the results is a combination

TABLE 6.2

Numerical results of $f''(\eta)$ at $\eta = 1$ for various values of R compared with the analytic results obtained from equation (6.2) together with the exponentially small correction (6.3)

R	*Section*	$f''(1)$ *from* (6.2)	*Correction* (6.3)	*Corrected* $f''(1)$	*Numerical*
28.142	I	−26.840	1.444	−25.396	−25.502
37.519	I	−36.306	0.334	−35.972	−35.959
49.098	I	−47.942	0.041	−47.901	−47.893
55.934	I	−54.800	0.011	−54.789	−54.784
65.674	I	−64.562	0.001	−64.561	−64.558
28.139	II	−26.837	−1.445	−28.282	−28.399
37.791	II	−36.579	−0.319	−36.898	−36.884
44.717	II	−43.543	−0.093	−43.636	−43.625
56.008	II	−54.874	−0.011	−54.885	−54.880
64.296	II	−63.182	−0.002	−63.184	−63.180

of the above error together with the error in the correction (6.3), which is due to the exclusion of higher order terms in the first series and also the omission of the terms of $0(\gamma^2)$. When $R < 30$ the error is almost completely due to the terms of $0(\gamma^2)$.

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